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LOWER BOUND ON DEFORMATION FOR DYNAMICALLY LOADED RIGID-PLASTIC STRUCTURES

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Abstract—A lower bound on maximum deformation is determined for rigid-plastic structures subjected to time dependent loads. This bound on deformation amalgamates and slightly extends two previous bounds. It is easily calculated based on an assumed velocity field that is kinematically admissible. Comparisons are made between this bound, the complementary upper bound by Robinson[5], and the analytical solution for maximum deformation of five different structural elements. Thus, characteristics of the structure and applied tractions that affect accuracy of the bound are examined. In two cases, the stress field transition from bending at small deformations to membrane stresses at large deformation is demonstrated.

INTRODUCTION

Despite the simplification of the rigid-plastic material idealization, analytical solutions for plastic deformation in dynamically loaded structural elements have been found for only a few problems. Two approaches have been used to estimate deformations for a broader class of problems: modal approximation and bounding theorems. Both approaches simplify calculations by assuming a time independent, kinematically admissible displacement field or statically admissible load. Although nodal approximations are frequently closer to the analytical solution, the certainty of bounding theorems may be preferred. This is particularly so when both upper and lower bounds can be easily calculated.

An upper bound theorem on maximum displacement and a lower bound theorem on response time for impulsively loaded, rigid-plastic structures were first obtained by Martin[1, 2]. This displacement bound on a selected point in the body is in terms of the initial kinetic energy and a statically admissible field of surface tractions and body forces. Ploch and Wierzbicki[3] extended the range of applicability of this bound to include large deformations. For cases of time dependent rather than impulsive loads, a lower bound on response time and an upper bound on displacement were derived by Kaliszky[4] and Robinson[5] respectively. Robinson's displacement bound has had limited use since it requires the analytical solution to the problem during the time that tractions are non-zero.

A complementary lower bound theorem on maximum displacement was obtained by Morales and Neville[6] for the case of impulsive loads. This lower bound was clarified by Wierzbicki[7, 8] who also commented on its usefulness with an arbitrary distribution of initial velocity.

The present investigation slightly extends a lower bound on maximum final displacement for time-dependent tractions with arbitrary spatial distributions by Morales[6, 9]. In terms of applied load, this bound is complementary to Robinson's upper bound. This lower bound on maximum displacement is compared with several exact solutions from beam and plate deformation analyses. As part of the derivation of the displacement bound, a lower bound on structural response time is also determined.

PROBLEM FORMULATION

A body with volume V is composed of a rigid-plastic material with density ρ. The body, which is moving with an initial velocity distribution $v_i^0(x_j)$, is subjected to time-dependent tractions T_i on the surface S. Body forces are assumed to be negligible.

The body deforms when tractions and initial velocity are sufficiently large that stresses across some cross section equal the yield stress. Let $t = 0$ be the time this structural yield condition is first satisfied. The response of the body at any time $t > 0$ may be characterized by time dependent velocity and acceleration fields $\dot{u}_i$, $\ddot{u}_i$ and the associated time dependent stress and strain-rate fields σ_{ij}, $\dot{\epsilon}_{ij}$. The strain-rate field is related to the yield function $\phi(\sigma_{ij})$ through the flow rule

$$\dot{\epsilon}_{ij} = \lambda\langle\phi(\sigma_{ij})\rangle \partial\phi/\partial\sigma_{ij} \tag{1}$$

where a rigid-plastic material is defined by

$$\begin{aligned} \langle\phi(\sigma_{ij})\rangle &= 0 \ \text{ for } \ \phi(\sigma_{ij}) < 0 \\ \langle\phi(\sigma_{ij})\rangle &= 1 \ \text{ for } \ \phi(\sigma_{ij}) = 0 \end{aligned} \tag{2}$$

and $\phi(\sigma_{ij}) > 0$ is not admitted. Convexity of the yield function and normality of the plastic strain-rate vectors $\dot{\epsilon}_{ij}^p$ to the yield surface $\phi = 0$ are consequences of Drucker's postulate for stable materials[9]

$$(\sigma_{ij}^p - \sigma_{ij})\dot{\epsilon}_{ij}^p \geqslant 0 \tag{3}$$

where σ_{ij}^p is any stress state satisfying $\phi(\sigma_{ij}^p) = 0$ while $\phi(\sigma_{ij}) < 0$. During any motion $\dot{u}_i^p$ resulting in σ_{ij}^p somewhere in the body, plastic deformation dissipates energy. Define an energy dissipation rate $D(\dot{u}_i^p)$ for the body. Then

$$D(\dot{u}_i^p) = \int_V \sigma_{ij}^p \dot{\epsilon}_{ij}^p \,\mathrm{d}V \geqslant \int_V \sigma_{ij}\dot{\epsilon}_{ij}^p \,\mathrm{d}V. \tag{4}$$

The inequality results from the stability postulate.

Suppose one assumes some kinematically admissible velocity field $\dot{u}_j^*$ and define σ_{ij}^*, $\dot{\epsilon}_{ij}^*$ as the stress and strain-rate fields associated with the assumed velocity field. Then, the principle of virtual velocities results in

$$\int_S T_i \dot{u}_i^* \,\mathrm{d}S - \int_V \sigma_{ij}\dot{\epsilon}_{ij}^* \,\mathrm{d}V = \int_V \rho \ddot{u}_i \dot{u}_i^* \,\mathrm{d}V \tag{5}$$

where at any instant of time, the stress and acceleration fields in the body σ_{ij}, $\ddot{u}_i$ are in equilibrium with the applied tractions T_i. Since the assumed stress and strain-rate fields σ_{ij}^*, $\dot{\epsilon}_{ij}^*$ satisfy eqns (1) and (2) for a rigid-plastic material, the assumed velocity field $\dot{u}_i^*$ satisfies the basic inequality for total energy dissipated during the time of motion, t_f; hence

$$\int_0^{t_f} D(\dot{u}_i^*)\,\mathrm{d}t \geqslant \int_S \mathrm{d}S \int_0^{t_f} T_i\dot{u}_i^* \,\mathrm{d}t - \int_V \mathrm{d}V \int_0^{t_f} \rho\ddot{u}_i\dot{u}_i^* \,\mathrm{d}t. \tag{6}$$

LOWER BOUND ON RESPONSE TIME

It will be shown that a structure subjected to dynamic loads has a response time that is longer than the response time of the same structure in an assumed deformation mode. Consider an assumed velocity field that is independent of time $\dot{u}_i^*(x_j, t) = v_i^*\psi^*(x_j)$ where $\psi^* \leq 1$. Then $D(\dot{u}_i^*)$ is also independent of time when strain-rate and deformation rate are linearly related.† After integrating the last term in eqn (6) by parts

$$t_f D(\dot{u}_i^*) \geqq \int_S \mathrm{d}S \int_0^{t_f} T_i\dot{u}_i^* \,\mathrm{d}t + \int_V \rho v_i^0 \dot{u}_i^* \,\mathrm{d}V \tag{7}$$

where $v_i^0(x_j)$ is the initial velocity field imparted by any impulsive load. This integration requires that the direction cosines of the actual velocity field remain constant during the deformation process. Now, if the pressure pulse duration is τ, compact pulses are defined as having $\tau \leqq t_f$.

†Generally, when $D(\dot{u}_i^*)$ is not independent of time, the largest energy dissipation rate occuring during a deformation must be used.

Noting that the tractions $T_i(t)$ only act during $t \leqq \tau$, the lower bound on response time to compact pulses will be

$$t_f{}^* = \frac{1}{D(\dot{u}_i{}^*)}\left\{\int_S \mathrm{d}S \int_0^\tau T_i\dot{u}_i{}^*\,\mathrm{d}t + \int_V \rho v_i{}^0\dot{u}_i{}^*\,\mathrm{d}V\right\}. \tag{8}$$

Kaliszky[4] obtained an equivalent lower bound on response time for tractions that are separable functions $T_i = p(t)T_i^0(x)$. However, the restriction of separability has not been required in this proof†.

When the body is initially at rest, what is required are tractions that are larger than a static distribution of traction that will cause structural collapse in the assumed deformation mode. Call this static collapse loading T_i^s. Consequently, when $v_i^0 = 0$, this bound applies to timewise compact pulses where $T_i > T_i^s$.

An alternative expression for the lower bound on structural response time can be developed from considerations on the rate of work done by the static collapse force T_i^c, for the assumed velocity field. Let T_i^s be a statically admissible force at each point of maximum velocity in the assumed velocity field ($\psi = 1$). T_i^s is coincident with the maximum velocity vector, $v_i{}^*$. The corresponding stress field, σ_{ij}^s may have isolated regions where $\phi(\sigma_{ij}^s) = 0$ but there is no mechanism so collapse does not occur. Hence, $\ddot{u}_i^s = 0$ and the principle of virtual velocities gives

$$\int_S T_i^s\dot{u}_i{}^*\,\mathrm{d}S - \int_V \sigma_{ij}^s\dot{\epsilon}_{ij}{}^*\,\mathrm{d}V = 0. \tag{9}$$

Subtracting eqn (9) from (5)

$$\int_S (T_i - T_i^s)\dot{u}_i{}^*\,\mathrm{d}S - \int_V \rho\ddot{u}_i\dot{u}_i{}^*\,\mathrm{d}V = \int_V [(\sigma_{ij}{}^* - \sigma_{ij}^s) - (\sigma_{ij}{}^* - \sigma_{ij})]\dot{\epsilon}_{ij}{}^*\,\mathrm{d}V. \tag{10}$$

Now consider the stress state as the statically admissible tractions approach the static collapse force for the assumed deformation mode. In regions where $|\dot{\epsilon}_{ij}{}^*| > 0$, $\lim_{T_i^s \to T_i^c} \sigma_{ij}^s = \sigma_{ij}{}^*$. Consequently, the stability postulate (3) results in an inequality,

$$\int_S (T_i - T_i^c)\dot{u}_i{}^*\,\mathrm{d}S - \int_V \rho\ddot{u}_i\dot{u}_i{}^*\,\mathrm{d}V \leqq 0. \tag{11}$$

This equation can be easily integrated with respect to time after $\dot{u}_i{}^*$ is specified. When the assumed velocity field is independent of time, this gives a lower bound on response time.

$$t_f \geqq t_f{}^* \tag{12}$$

where

$$t_f{}^* = \frac{1}{T_i^c v_i{}^*}\left\{\int_S \mathrm{d}S \int_0^\tau T_i\dot{u}_i{}^*\,\mathrm{d}t + \int_V \rho\dot{u}_i{}^0\dot{u}_i{}^*\,\mathrm{d}V\right\}. \tag{13}$$

A time independent assumed velocity field and the associated static collapse force have the same stress field in the region where $|\dot{\epsilon}_{ij}{}^*| > 0$. Hence, by eqn (9) $T_i^c v_i{}^* = D(\dot{u}_i{}^*)$. The bound on response time is identical with eqn (8) whenever strain-rates are only proportional to deformation rates.

†Ting, Lee, Mukherjee and Nystrom previously derived this bound for time dependent tractions that are not separable functions[10].

LOWER BOUND ON MAXIMUM DISPLACEMENT

Morales found a lower bound on maximum displacement by using the principle of virtual velocities on an assumed, kinematically admissible velocity field[9]. Let $\dot{u}_i^*$ be a velocity field that decreases linearly with time during t_f^*,

$$\dot{u}_i^*(x, t) = \begin{cases} \dot{u}_i^{0*}(1 - t/t_f^*), & t < t_f^* \\ 0, & t > t_f^* \end{cases} \tag{14}$$

where the initial velocities are $\dot{u}_i^{0*} = v_i^{0*}\psi^*(x_i)$†. From eqn (5), the total energy dissipated in this deformation mode during t_f will be

$$\int_0^{t_f} \mathrm{d}t \int_V \sigma_{ij}\dot{\epsilon}_{ij}^* \,\mathrm{d}V = \int_0^{t_f} \mathrm{d}t \int_S T_i\dot{u}_i^{0*}(1 - t/t_f^*)\,\mathrm{d}S - \int_0^{t_f} \mathrm{d}t \int_V \rho u_i \dot{u}_i^{0*}(1 - t/t_f^*)\,\mathrm{d}V. \tag{15}$$

After interchanging the order of integration, the last term can be integrated by parts. Noting that $\dot{u}_i^* = 0$ for $t > t_f^*$

$$\frac{1}{t_f^*}\int_V \rho\dot{u}_i^{0*}u_i(x, t_f)\,\mathrm{d}V = \int_0^{t_f^*} \mathrm{d}t \int_S T_i\dot{u}_i^{0*}(1 - t/t_f^*)\,\mathrm{d}S + \int_V \rho\dot{u}_i^{0*}\dot{u}_i^0\,\mathrm{d}V - \int_0^{t_f^*} \mathrm{d}t \int_V \sigma_{ij}\dot{\epsilon}_{ij}^{0*}(1 - t/t_f^*)\,\mathrm{d}V. \tag{16}$$

The mean value theorem for integrals and the energy dissipation rate inequality (4) can be used in this equation to obtain the lower bound on the largest final displacement

$$\delta_i \int_V \rho\dot{u}_i^{0*}\,\mathrm{d}V \geqq \int_V \rho\dot{u}_i^{0*}u_i(x, t_f)\,\mathrm{d}V \geqq t_f^*\left\{\int_0^{t_f^*} \mathrm{d}t \int_S T_i\dot{u}_i^{0*}(1 - t/t_f^*)\,\mathrm{d}S + \int_V \rho\dot{u}_i^{0*}\dot{u}_i^0\,\mathrm{d}V - t_f^* D(\dot{u}_i^*)/2\right\}. \tag{17}$$

Then $\delta = \max[\delta_i]$ where

$$\delta_i = \max[u_i(x, t_f)], \quad x \in V. \tag{18}$$

Independent bounds corresponding to different components of displacement result from a set of assumed velocity fields $\dot{u}_i^*$ that each involve a separate velocity component [11].

This displacement bound is the same as that derived by Wierzbicki[8] for impulsive loads where $T_i = 0$ during $t > 0$ and by Morales[9] for time-dependent tractions. It is not necessary for the initial impulse to be uniformly distributed. The bound is satisfactory for any arbitrary distribution of traction or initial velocity.

The lower bound on maximum displacement can be expressed more concisely as

$$\delta \geqq \frac{t_f^*}{2\int_V \rho\dot{u}_i^{0*}\,\mathrm{d}V}\left\{\int_0^{t_f^*} \mathrm{d}t \int_S (1 - 2t/t_f^*)T_i\dot{u}_i^{0*}\,\mathrm{d}S + \int_V \rho\dot{u}_i^0\dot{u}_i^{0*}\,\mathrm{d}V\right\} \tag{19}$$

by using the bound on response time, eqn (8).

The difference between this lower bound and the analytical solution generally depends on the number of spatial co-ordinates of the deformation field. The results from the use of the mean value theorem. (Maximum and mean displacements are closer in one dimensional deformation than they are in two dimensional deformation.)

†This velocity field results in the largest lower bound of the general class of time dependent functions, $\dot{u}_i^* = \dot{u}_i^{0*}(1 - t/t_f^*)^n$ where $n \geqq 1$ [23].

APPLICATIONS

Simply supported beam under uniform pressure

A simply supported beam of length $2L$ and mass m per unit length deforms under a uniformly distributed pressure $p(t)$. Lower bounds on the final centre displacement will be obtained for rectangular, triangular and exponentially decreasing pressure pulses.

The assumed velocity field is piecewise linear and symmetric about a centre plastic hinge, that is

$$\dot{u}^* = v^* x/L \text{ for } 0 \leq x \leq L. \tag{20}$$

Hence, for a beam with yield moment M_0

$$D = 2M_0 v^*/L \tag{21}$$

The response time and final displacement lower bounds, t_f^* and δ^*, calculated for three pressure pulse shapes are shown in Table 1. In this table, $\lambda = P/P_0$ is the ratio of peak pressure to static collapse pressure, τ is the characteristic time of pulse duration and

$$T_0 = \frac{1}{P}\int_0^\infty p \, dt \tag{22}$$

is a mean traction duration time that may be used to eliminate most effects of pulse shape[12, 13]. The analytical solutions for flexural deformation shown in Table 1 are for peak pressures $\lambda > 3$ where the rigid-plastic material model is most appropriate.

Figure 1 illustrates the same comparisons. The compact set of analytical solutions resulting from nondimensionalization using T_0 is compared with the upper and lower displacement bounds. The upper bound is determined by assuming the total impulse is applied impulsively (at $t = 0$).

Clamped beam under central impulsive load

A clamped rigid-plastic beam of length $2L$ and mass m per unit length is accelerated by an impulsive pressure acting over a central segment of length $2b$. This segment is given an initial transverse velocity v^0 while the remainder of the beam remains at rest. The ends of the beam are clamped to prevent both rotational and axial displacement. Bounds related to flexural and membrane deformations will be calculated.

Assuming a piecewise linear velocity field that is symmetric about a centre plastic hinge,

$$\dot{u}^* = v^* x/L \text{ for } 0 < x < L. \tag{23}$$

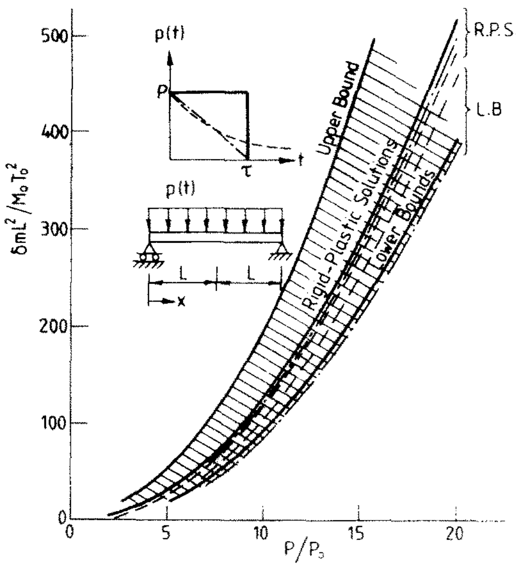
Hence, the dissipation rate at the centre hinge and the two end hinges will be

$$D^* = 4M_0 v^*/L \tag{24}$$

Table 1. Comparison of simply supported beam displacement bounds with high pressure solutions for three pulse shapes

Pressure	Eff. Load time T_0	Response time t_f^*	Lower displacement bound $\delta^* mL^2/M_0T_0^2$	Displacement $\delta mL^2/M_0T_0^2$	Upper displ. bound $\delta^{**} mL^2/M_0T_0^2$
$P = \begin{cases} P, & t < 1 \\ 0, & t > 1 \end{cases}$	τ	$\lambda\tau$	$\lambda(\lambda - 1)$	$\frac{4\lambda}{3}(\lambda - \frac{3}{4})$	$2\lambda^2$
$P = \begin{cases} P(1 - t/1), & t < \tau \\ 0, & t > \tau \end{cases}$	$\frac{\tau}{2}$	$\frac{\lambda\tau}{2}$	$\frac{\lambda}{3}(3\lambda - 4)$	$\frac{4\lambda}{3}(\lambda - 1)$	$2\lambda^2$
$P = Pe^{-t/\tau}$	τ	$2\lambda\tau(1 - e^{-1})$	$2\lambda[\lambda(1 + e^{-t_f^*/\tau})(1 - e^{-1}) - 1]$	†	$2\lambda^2$

†The analytical solution is lengthy and requires numerical evaluation.



BEAM DEFORMATION BOUNDS FOR THREE PULSE SHAPES

Fig. 1.

where M_0 is the yield moment for the beam. The bound on response time is

$$t_f{}^* = \frac{mv^0L^2}{4M_0}\frac{b}{L}\left(2-\frac{b}{L}\right). \tag{25}$$

A lower bound for the centre displacement calculated with the assumed velocity field (23) is

$$\delta_f{}^* = \frac{mb^2(v^0)^2}{8M_0}\left(2-\frac{b}{L}\right)^2. \tag{26}$$

A corresponding upper bound calculated by Martin's Theorem is

$$\delta_f^{**} = \frac{mbL(v^0)^2}{16M_0} \tag{27}$$

whereas the flexural solution for final centre displacement[14] is

$$\delta_f = \begin{cases} \dfrac{mb^2(v^0)^2}{3M_0}\left[\dfrac{5}{4}+\ln\left(\dfrac{L}{2b}\right)\right], & \dfrac{b}{L}<\dfrac{1}{2} \\ \dfrac{mb^2(v^0)^2}{12M_0}\left[\dfrac{6L}{b}-\dfrac{L^2}{b^2}-3\right], & \dfrac{b}{L}\geqslant\dfrac{1}{2}. \end{cases} \tag{28}$$

(This solution requires $b > H$, the beam height.) These bounds and the solution for the centre displacement of a clamped beam with an impulsive velocity over a central region are illustrated in Fig. 2.

In a clamped, rigid-plastic beam of rectangular cross-section, the "plastic hinge" phase of motion ends when $\delta/H = 1$. A "plastic-string" phase of motion describes larger deflections where extension is predominant over flexure. In this phase of motion the entire cross-section of the beam is in tension. The resultant force is the fully plastic yield force, N_0.

A lower displacement bound for this phase of motion can be calculated by again assuming the piecewise linear velocity field in eqn (23).† Since the centerline strain-rate is $\dot{\epsilon} = u\dot{u}/L^2$ in

†If the assumed velocity distribution is sinusoidal rather than linear, a smaller value is calculated for the lower displacement bound.

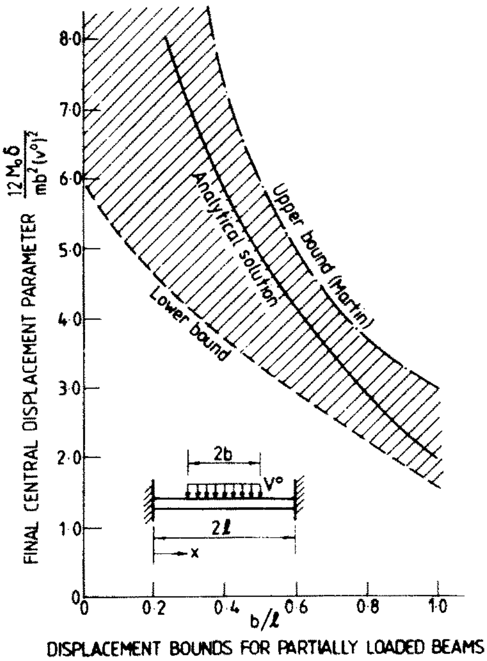


Fig. 2.

this deformation mode, the largest dissipation rate is

$$D(\dot{u}_i^*) = 2N_0v^*u^*(t)/L \leqslant 2N_0v^*\delta/L. \tag{29}$$

An expression for the lower bound on response time is therefore

$$t_f^* = mv^0bL(2-b/L)/2N_0\delta \tag{30}$$

which cannot be evaluated until a solution is obtained for δ. A lower bound for the centre displacement is

$$\delta^2 \geqslant \frac{mb(v^0)^2}{4N_0}\left(2-\frac{b}{L}\right)^2. \tag{31}$$

In Fig. 3 this displacement bound is compared with the solution for centre deflection of a clamped beam subject to a uniformly distributed initial velocity[15]. The parameter β generally depends on properties of the beam cross-section. For rectangular beams, $\beta \approx L/H$. The transformation from a bending to a string lower bound occurs at values of $\delta/H \approx 1$ (the correct lower bound is the least lower bound). Consequently, the lower bound on response time will be given by eqn (25) for $\delta/H < 1$ and by eqn (30) for larger deformations.

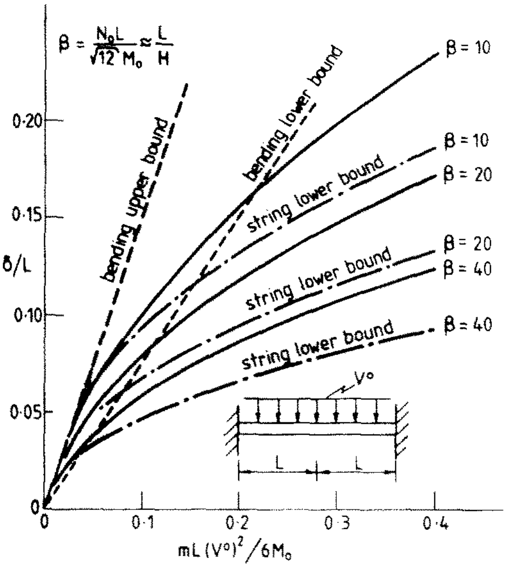
Cantilever beam with tip mass

A cantilever beam of length L and mass m per unit length is impacted at the tip by a mass G that adheres to the beam. G impacts with velocity v^0 transverse to the axis of the beam. Both upper and lower bounds on the final tip displacement will be calculated for this example of a concentrated impulsive load.

Assume a velocity field corresponding to a hinge at the base of a rigid beam

$$\dot{u}^* = v^*x/L. \tag{32}$$

When the beam has a yield moment M_0, the energy dissipation rate for the assumed velocity



CENTER DISPLACEMENT OF IMPULSIVELY LOADED BEAM

Fig. 3.

field is

$$D = M_0 v^*/L. \tag{33}$$

The bound on response time for the impulsive load is therefore

$$t_f^* = GLv^0/M_0. \tag{34}$$

The lower bound on tip displacement calculated for the assumed velocity field is

$$\delta^* = GL(v^0)^2/2M_0(1+\gamma) \tag{35}$$

where $\gamma = mL/2G$ is half the beam-to-impacting mass ratio. The beam will collapse under static tip forces larger than $F^s = M_0/L$ so Martin's upper bound on tip displacement is $\delta^{**} = GL(v^0)^2/2M_0$. Consequently,

$$1 \leq 2M_0\delta/GL(v^0)^2 \leq (1+\gamma)^{-1}. \tag{36}$$

An analytical solution to this problem by Parkes[16] involves a hinge that starts in the interior of the beam and moves to the base. The final tip displacement is

$$2M_0\delta/GL(v^0)^2 = [(1+\gamma)^{-1} + 2\gamma^{-1}\ln(1+\gamma)]/3. \tag{37}$$

These bounds and the analytical solution are compared as a function of the mass ratio in Fig. 4. When the beam has negligible mass in comparison with G, the assumed and analytical deformation modes are the same. Hence, at $\gamma = 0$ both bounds converge to the analytical solution. Since Martin's upper bound is independent of the beam mass in this problem, it is not as close to the solution as the lower bound for large values of γ.

Circular plate under uniformly distributed pressure

A circular plate of radius R and mass m per unit area is accelerated by a pressure $p(t)$ distributed uniformly within a central circle of radius a. Lower bounds on the displacement will be obtained for simply supported and clamped boundary conditions.

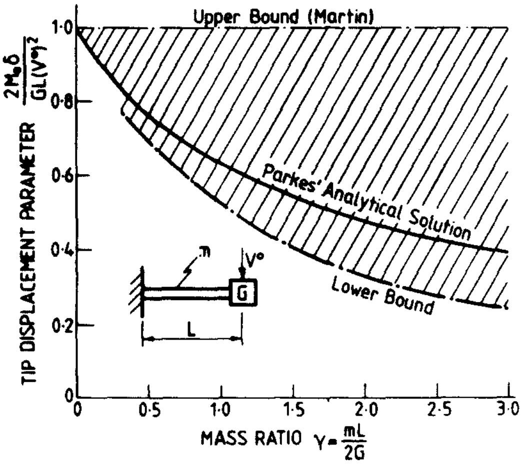


Fig. 4.

The assumed velocity field is piecewise linear and symmetric about the centre.

$$\dot{u}^* = v^*(R-r)/R \quad \text{for} \quad 0 \leq r \leq R. \tag{38}$$

The associated curvature rates for $r < R$ are

$$\dot{\kappa}_r^* = -\partial^2 u^*/\partial r^2 = 0, \quad \dot{\kappa}_\theta^* = -r^{-1}\partial u^*/\partial r = v^*/rR. \tag{39}$$

With a Tresca type yield condition this implies $M_r < M_0$, $M_\theta = M_0$ so the energy dissipation rate for a simply supported plate is

$$D = 2\pi M_0 v^*. \tag{40}$$

For a rectangular pressure pulse, the bound on response time will be

$$t_f^* = P\tau a^2(3-2a/R)/6M_0. \tag{41}$$

Thus, the lower bound on centre displacement is

$$\delta \geq \frac{(P\tau a^2/R)^2}{12\, mM_0}\left(3-\frac{2a}{R}\right)\left(3-\frac{2a}{R}-\frac{6M_0}{Pa^2}\right). \tag{42}$$

When the edge of the plate is clamped rather than simply supported a yield hinge exists at the edge where $\dot{\kappa}_r = v^*/R$. In this case, $M_r = M_0$ at $r = R$ and the energy dissipation rate for a clamped plate is

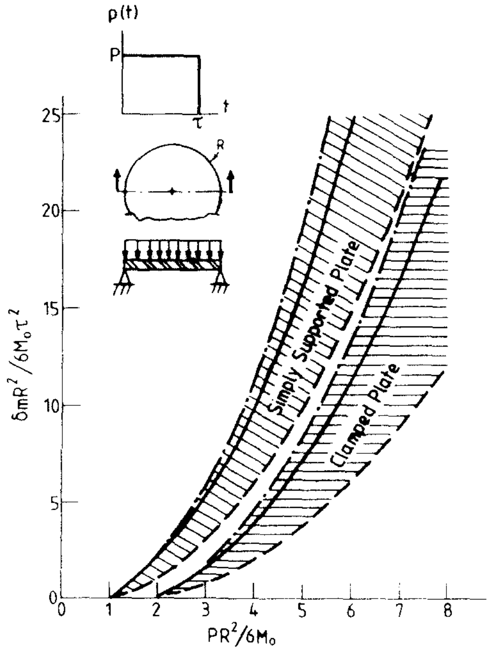
$$D = 4\pi M_0 v^*. \tag{43}$$

The energy dissipation at the edge and in the interior of the plate are the same. The bounds on response time to a rectangular pressure pulse are

$$t_f^* = (P\tau a^2)(3-2a/R)/12M_0 \tag{44}$$

resulting in a bound on the centre displacement

$$\delta \geq \frac{(P\tau a^2/R)^2}{24mM_0}\left(3-\frac{2a}{R}\right)\left(3-\frac{2a}{R}-\frac{12M_0}{Pa^2}\right). \tag{45}$$



CENTER DISPLACEMENT OF UNIFORMLY LOADED CIRCULAR RIGID-PLASTIC PLATE

Fig. 5.

In Fig. 5 these lower bounds and Robinson's upper bounds for the final centre displacement of simply supported and clamped plates are compared with the analytical solutions by Hopkins and Prager[17] and Florence[18] respectively. Both bounds have the proper functional behavior and the correct origin at $\pi R^2/PF_i^s = 1$. In this case, the upper bound is closer to the analytical solution. This is not surprising since Robinson's upper bound uses the analytical solution for the early time, moving hinge phase of motion, whereas the lower bound was calculated from a velocity field that was assumed to be linear.

When the rectangular pressure pulse acts only on the central area of the plate, additional stages can arise in the deformation process. The lower bound on deformation does not depend upon these details and consequently, the assumed and actual velocity fields can be quite different. Figure 6 compares these bounds with an analytical solution for a clamped plate by Florence[19]. The lower bound is approximately half the analytical result. A distributed static collapse pressure for a clamped plate $P_0 = 4M_0/a^2(1-2a/3R)$ has been used to obtain a ratio of peak to static collapse pressure, P/P_0.

Annular plate response to transverse pressure pulse

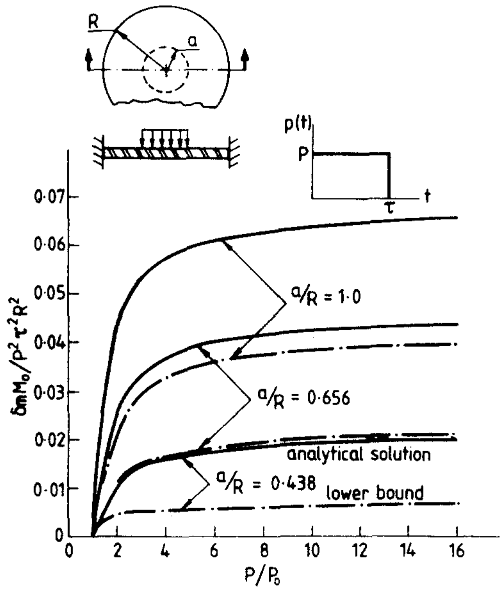
A simply supported circular plate containing a central hole with radius "a", is subjected to a temporally rectangular pressure pulse. Spatially, the pressure increases from zero at the supported edge to P at the hole. Deflections of this plate are considerably reduced by membrane forces, even when they are less than the plate thickness. Bounds related to both flexural and membrane deformations will be compared with an analytical solution by Jones[20].

The velocity field is assumed to decrease linearly from the inner to the outer edge; that is

$$\dot{u}_i^* = v^*(R-r)/(R-a). \tag{46}$$

The associated curvature rates for $r < R$ are

$$\dot{\kappa}_r^* = -\partial^2 u/\partial r^2 = 0,\ \dot{\kappa}_\theta^* = -r^{-1}\partial u^*/\partial r = v^*/r(R-a) \tag{47}$$



CENTER DISPLACEMENT OF CENTRALLY LOADED CIRCULAR RIGID-PLASTIC PLATE

Fig. 6.

so $M_r < M_0$ and $M_\theta = M_0$. The energy dissipation rate for this flexural mode of deformation is

$$D = 2\pi M_0 v^* \tag{48}$$

which is the same as that of a full circular plate. For a distributed pressure

$$p(r, t) = \begin{cases} P(R-r)/(R-a) & t < \tau \\ 0 & t > \tau. \end{cases} \tag{49}$$

A bound on response time is

$$t_f^* = P\tau/P_0 \tag{50}$$

where the static collapse pressure for the plate with this pressure distribution is $P_0 = 6M_0/(R-a)(R+2a)$ and the static collapse force distributed around $r = a$ is $T_i^s = 2\pi M_0$. When the mass per unit area is m, the lower bound on final displacement at the inner edge will be

$$\delta \geqslant \frac{3M_0P^2\tau^2(1-P_0/P)}{mP_0^2(R-a)(R+2a)}. \tag{51}$$

A membrane, rather than flexural, mode of deformation can be considered using the same conical assumed velocity field as in eqn (46). When u_a denotes the transverse displacement at the inner edge, the in-plane strain-rate components associated with this velocity field will be

$$\dot{\epsilon}_r^* = \dot{\epsilon}_{r\theta}^* = 0, \quad \dot{\epsilon}_\theta^* = u_a^*\dot{u}_a^*(R-r)/r(R-a)^2. \tag{52}$$

For a plate of uniform thickness H, and a yield stress σ_y, the energy dissipation rate resulting from this circumferential strain-rate field is

$$D = \pi\sigma_y H u_a^* \dot{u}_a^* \leqslant \pi\sigma_y H v^* \delta. \tag{53}$$

Recalling that $M_0 = \sigma_y H^2/4$, the lower bound on response time to the rectangular pressure pulse in eqn (49) is

$$t_f^* = P\tau H/2P_0 u_a^*. \tag{54}$$

The lower bound on final deformation at the inner edge for this membrane mode of deformation is

$$\delta \geq \frac{P\tau^2}{4m}\left[\left(1+\frac{4mH}{P_0\tau^2}\right)^{1/2} - 1\right]. \tag{55}$$

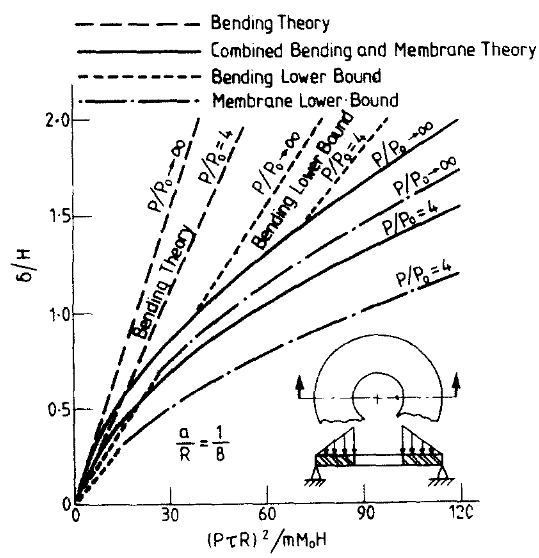
Both the bending and membrane lower bounds on maximum deformation are compared with the analytical solution by Jones in Fig. 7. The additional stiffness due to membrane forces is evident in this figure. Membrane forces dominate at deflections larger than the plate thickness. Whereas bounds on the bending solution are half the analytical result, bounds on the membrane solution are much closer. This accuracy results from the assumed mode of deformation being a better approximation to the deforming configuration during motion of the plate.

Cylindrical shell response to radial pressure pulse

A rigid-plastic shell of radius R, thickness H, length $2L$ and mass per unit area m is subjected to a radial pressure $p(t)$. Both simply supported and clamped end conditions with axial constraints are considered. Deformations of this shell have been determined by Hodge for several pressure pulse shapes[21]. His analysis applies to small deflections. When deflections become large in comparison with the shell thickness, both bending and membrane forces are important. For the case of impulsive pressures and short shells, Jones showed that the shell deformation is considerably reduced by membrane forces[22]. A lower bound on the shell deformation in the range of large deflections will be calculated and compared with these analytical solutions.

A lower bound on maximum deformation can be calculated by assuming a radial velocity field that is symmetric about the centre and increasing linearly from the ends,

$$\dot{u}^* = v^* x/L \quad \text{for} \quad 0 < x < L. \tag{56}$$



ANNULAR PLATE DEFORMATION BY RADIALLY DECREASING RECTANGULAR PRESSURE PULSE

Fig. 7.

This velocity field results in a yield circle circumscribing the shell at the centre where the curvature rate is $\dot{\kappa}_x = 2v^*/L$. In the case of clamped ends, there are also yield circles around each end of the shell where the curvature rate is $\dot{\kappa}_x = v^*/L$. (These yield circles correspond to yield hinges in a plane structure.) Throughout the shell the axial and circumferential inplane strain rates are

$$\dot{\epsilon}_x = u^*v^*/L^2,\ \dot{\epsilon}_\theta = xv^*/RL. \tag{57}$$

For a yield surface that involves no interaction between stress components, these strain rates result in an energy dissipation rate

$$D = 4\pi Rv^*\left[N_0\left(\frac{u^*}{L}+\frac{L}{2R}\right)+\frac{\alpha M_0}{L}\right] \tag{58}$$

where $\alpha = 1$ for simply supported ends and $\alpha = 2$ for clamped ends. The response time of the cylindrical shell to rectangular pressure pulses of duration τ is bounded below by

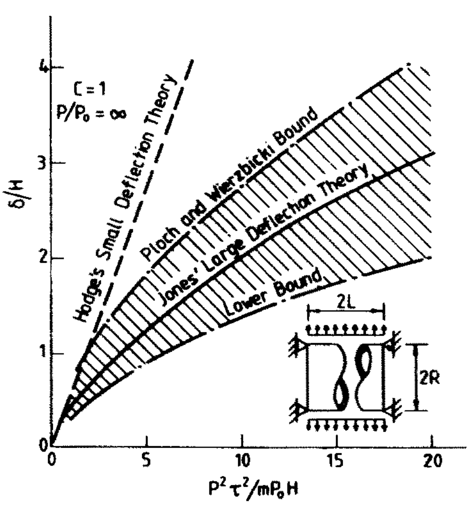
$$t_f^* = \frac{P\tau(\alpha+c)}{P_0(\alpha+c+4\delta/H)} \tag{59}$$

where the static collapse pressure is $P_0 = (1+\alpha/c)N_0/R$ and $c = 2L^2/RH$ is a parameter describing the shell geometry.

Rigid-plastic structural deformation is generally a function of applied impulse squared. Hence, a loading parameter for the rectangular pressure pulse is defined as $\psi = P^2\tau^2/mP_0H$ where P is the pressure magnitude. Using this parameter, the lower bound on radial displacement at the centre of the shell can be expressed as

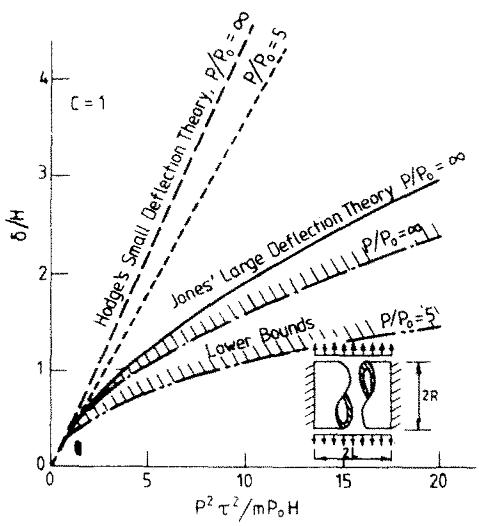
$$\frac{\delta^*}{H} = \frac{(\alpha+c+2P_0\psi/P)}{8}\left\{-1+\left[1+\frac{8\psi(\alpha+c)(1-P_0/P)}{(\alpha+c+2P_0\psi/P)^2}\right]^{1/2}\right\}. \tag{60}$$

In Fig. 8 this lower bound and an upper bound by Ploch and Wierbicki[3] are compared with the small and large displacement analytical solutions for impulsively loaded, simply supported, shells. When the shell is clamped rather than simply supported, plastic deformation involves hinge circles at the ends. For the same loading impulse, the additional energy dissipated at end hinge circles causes a clamped shell to have less deformation than a simply supported shell. In



SIMPLY SUPPORTED CYLINDRICAL SHELL DEFORMATION BY IMPULSIVE PRESSURE

Fig. 8.



CLAMPED CYLINDRICAL SHELL DEFORMATION BY RECTANGULAR PRESSURE PULSES

Fig. 9.

Fig. 9, lower bounds on plastic deformation of clamped beams are compared with analytical solutions for high and low pressure rectangular loading pulses. (Note that the loading parameter ψ depends on $P_0 = N_0(\alpha + c)/Rc$ so it is not apparent that for the same pulse, the deformation in Fig. 9 is larger than that in Fig. 8.) These comparisons have been made for short shells ($c = 1$) where the energy dissipated in bending and membrane deformations are comparable at $\delta/H = 1$. With long shells, the effect of end constraints on plastic deformation will be less significant.

CONCLUSION

A more concise and less restrictive expression for a lower bound on maximum final displacement resulting from time dependent tractions with arbitrary spatial distribution has been determined. Together with a complementary upper bound determined by Robinson[5], this provides an easily calculated method of obtaining limits on plastic deformation due to transient loads. The method has been used to analyse both small and large deformation structural response. Large deformations are brought into the analysis by considering the membrane stress distribution in the assumed deformation velocity field.

The displacement lower bound and the analytical solution have been compared for five different structural elements. Accuracy of the bound depends on how closely peak values of work rate done by applied tractions acting through the assumed velocity field represent the mean value. With uniformly distributed tractions and 1-dimensional deformation fields, the bound and the analytical solution are close. With concentrated forces and 2-dimensional deformation fields, the bound and the analytical solution are further apart.

REFERENCES

1. J. B. Martin, Impulsive loading theorems for rigid-plastic continua. *J. Engineer Mech. Div. ASCE*, **90**, *EM5*, 27–42 (1964).
2. J. B. Martin, A Displacement Bound Principle for Inelastic Continua Subjected to Certain Classes of Dynamic Loading. *J. of Applied Mech. ASME*, **31**, 1–6 (1965).
3. J. Ploch and T. Wierzbicki, Bounds for large plastic deformations of dynamically loaded continua and structures. *Int. J. Solids Structures* **17**, 183–195 (1981).
4. S. Kaliszky, Approximate solutions for impulsively loaded inelastic structures and continua. *Int. J. Non-Linear Mechanics* **5**, 143–158 (1970).
5. D. N. Robinson, A displacement bound principle for elastic–plastic structures subjected to blast loading. *J. Mech. Phys. Solids* **18**, 65–80 (1970).
6. W. J. Morales and G. E. Neville Jr., Lower bounds on deformation of dynamically loaded rigid-plastic continua. *AIAA J.* **8**(11), 2043–2046 (1970).
7. T. Wierzbicki, Comment on lower bounds on deformations of dynamically loaded rigid-plastic continua. *AIAA J.* **10**(3), 364–365 (1972).

8. T. Wierzbicki, Improved lower bound theorems for impulsively loaded continua. *Archives of Mechanics* **23**(3), 423–425 (1971).
9. W. J. Morales, Displacement bounds for blast loaded structures. *ASCE*, EM4, 965–974 (1972).
10. D. C. Drucker, A more fundamental approach to stress-strain relations. *Proc. 1st U.S. Natl. Cong. Appl. Mech.* 487–491 (1951).
11. T. C. T. Ting, E. H. Lee, S. Mukherjee and G. A. Nystrom, Methods of estimating impact duration and final deformation in collisions. *Stanford University Rep.* SUDAM 75-3 (1975).
12. P. S. Symonds, Large plastic deformations of beams under blast type loading. *Proc. 2nd U.S. Cong. Appl. Mech.* 505–515 New York, (1955).
13. C. K. Youngdahl, Correlation parameters for eliminating the effect of pulse shape on dynamic plastic deformation. *J. Appl. Mech.* **37**, 744–753 (1970).
14. J. B. Martin and P. S. Symonds, Mode Approximations for Impulsively Loaded Rigid-Plastic Structures. *J. Engng Mech. Div. ASCE* **92**, EM5, 43–66 (1961).
15. P. S. Symonds and T. J. Mentel, Impulsive loading of plastic beams with axial constraints. *J. Mech. Phys. Solids* **6**, 186–202 (1958).
16. E. W. Parkes, The permanent deformation of a cantilever struck transversely at its tip. *Proc. Roy. Soc.* **A228**, 462 (1955).
17. H. G. Hopkins and W. Prager, On the dynamics of plastic circular plates. *Z. angew. Math. Phys.* **5**, 317–330 (1954).
18. A. L. Florence, Clamped circular rigid-plates under blast loading. *J. Applied Mech.* **33** Series E, 256–260 (1966).
19. A. L. Florence, Clamped circular rigid-plastic plates under central blast loading. *Int. J. Solids Structures* **2**, 319–335 (1966).
20. N. Jones, Finite deflections of a simply supported rigid-plastic annular plate loaded dynamically. *Int. J. Solids Structures* **4**, 593–603 (1968).
21. P. G. Hodge, The influence of blast characteristics on the final deformation of circular cylindrical shells. *J. of Appl. Mech.* **23**, Trans ASME **78**, 617–624 (1956).
22. N. Jones, The influence of large deflections on the behaviour of rigid-plastic cylindrical shells loaded impulsively. *J. of Applied Mech.* **37**, 416–424 (1970).
23. Vu Van The and A. Sawczuk, Lower bounds to large displacements of impulsively loaded plastically orthotropic structures. *Int. J. Solids Structures* **19**, 189–205 (1983).